ON SUBSEQUENCES OF THE HAAR SYSTEM IN $L_p[0, 1]$, $(1 < p < \infty)$†

BY

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ABSTRACT

We show that if X is the closed linear span in $L_p[0,1]$ of a subsequence of the Haar system, then X is isomorphic either to l_p or to L_p [0,1], $[1 < p < \infty]$. We give criteria to determine which of these cases holds; for a given subsequence, this is independent of p.

The $\mathscr{L}_p$-spaces were introduced by J. Lindenstrauss and A. Pełczyński [5] as generalizations of $L_p(\mu)$ spaces; their unit-balls possess similar properties. In fact it has been shown that a separable B-space is a $\mathscr{L}_p$-space if and only if it is isomorphic to a complemented subspace of $L_p[0,1]$ but not isomorphic to l_2, $(1 < p \neq 2 < \infty)$. The separable $\mathscr{L}_2$-spaces are those which are isomorphic to l_2 (Lindenstrauss and Pełczyński [5], Lindenstrauss and Rosenthal [7].)

Thus, to find the isomorphic types of separable $\mathscr{L}_p$-spaces $(1 < p < \infty)$ we need to find the isomorphic types of complemented subspaces of $L_p[0,1]$. There are, at present, nine known different isomorphic types of separable $\mathscr{L}_p$-spaces $(1 < p \neq 2 < \infty)$, mostly due to H.P. Rosenthal [9].

It has been shown recently by W.B. Johnson, H.P. Rosenthal and M. Zippin [4] that the $\mathscr{L}_p$-spaces have bases. Hence the problem of finding the isomorphism types of separable $\mathscr{L}_p$-spaces is the same as finding the isomorphic types of complemented closed linear spans of subsequences of bases in $L_p[0,1]$, $(1 < p < \infty)$.

We give a partial result towards the solution of this problem.

Since the Haar basis is the most meager of all bases for $L_p[0, 1]$, this is a

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natural starting point (Lindenstrauss and Pełczyński [6]). It is unconditional so all subsequences yield complemented subspaces.

The Haar functions on the unit interval are usually defined as follows:

$$y_1(t) \equiv 1$$

and

$$y_{2^n+m}(t) = \begin{cases} 1 \text{ if } \dfrac{2m-2}{2^{n+1}} \leqq t < \dfrac{2m-1}{2^{n+1}}, \\ -1 \text{ if } \dfrac{2m-1}{2^{n+1}} \leqq t < \dfrac{2m}{2^{n+1}}, \\ 0 \quad \text{otherwise}, \end{cases}$$

for

$$n = 0 \cdots \text{ and } m = 1 \cdots 2^n.$$

It is standard to normalize the Haar functions so that the biorthogonal functional associated with the ith Haar function is simply integration with respect to the ith Haar function. For our purposes, however, it is more convenient to have these functions normalized with the supremum norm as above. In the following we will always use $\{y_i\}$ to mean the Haar functions.

It is well known that the Haar functions form an unconditional basis for $L_p[0,1]$ if $1 < p < \infty$. The proof is due to R.E.A.C. Paley and J. Marcinkiewicz and can be found in I. Singer [10].

We now state some well-known properties of unconditional basic sequences and some properties of l_p and $L_p[0,1]$ which we will use.

LEMMA 1. *If $\{x_i\}$ is an unconditional basic sequence and $\{x_{i_n}\}$ a subsequence, then the map P:*

$P(x_{i_n}) = x_{i_n}$, $n = 1 \cdots$ and $P(x_i) = 0$ for other indices, defines a projection of $[x_i]$ onto $[x_{i_n}]$. In fact an upper bound can be given for the norms of such projections from a particular unconditional basis.

LEMMA 2. *If $\{x_i\}$ is an unconditional basic sequence in a B-space X, if $[x_i]$ is complemented in X by a projection P, and if $\{z_i\}$ is a sequence in X with $\sum_{i=1}^{\infty} \|P\| \, \|x_i'\| \, \|x_i - z_i\| < 1$ where x_i' is the coefficient functional associated with x_i, then $\{z_i\}$ is an unconditional basic sequence, $[z_i]$ is complemented in X, and the correspondence $x_i \leftrightarrow z_i$ defines an isomorphism* [2].

LEMMA 3. *An infinite-dimensional complemented subspace of l_p is isomorphic to l_p $(1 < p < \infty)$* [8].

LEMMA 4. *Let $\{d_i\}$ be a sequence of $\{-1,0,1\}$ valued measurable functions and D their closed linear span in $L_p[0,1]$ for some p with $1 < p < \infty$.*

Define sets D_1 and $D_{n,m}$ for $n = 0 \cdots, m = 1 \cdots 2^{n+1}$ as follows:

$$D_1 = \text{support } d_1$$

$$D_{n,2m-1} = \text{support } d^+_{2^n+m}$$

$$D_{n,2m} = \text{support } d^-_{2^n+m}$$

Suppose support $d_1^- = \varnothing$

$$D_1 = D_{0,1} \cup D_{0,2}$$

$$D_{n,m} = D_{n+1,2m-1} \cup D_{n+1,2m}$$

and that there exist constants K_1 and K_2 with

$$0 < K_1 \leqq \mu(D_1) \leqq K_2 < \infty \text{ and } K_1 \leqq 2^{n+1}\mu(D_{n,m}) \leqq K_2.$$

Then the correspondence $Td_i = y_i$ between $\{d_i\}$ and the Haar system $\{y_i\}$ defines a lattice isomorphism between D and $L_p[0,1]$. Hence D forms a sublattice and is complemented in $L_p[0,1]$ by a projection of norm 1.

In addition

$$\|T\| \leqq (1/K_1)^{(1/p)}$$

$$\|T^{-1}\| \leqq K_2^{(1/p)}.$$

PROOF. It is easy to see from definitions that the unions

$$D_1 = D_{0,1} \cup D_{0,2}$$

$$D_{n,m} = D_{n+1,2m-1} \cup D_{n+1,2m}$$

are disjoint.

Thus if $\lambda_1 \cdots \lambda_k$ are constants, the functions $\Sigma\lambda_i d_i$ and $\Sigma\lambda_i y_i$ achieve the same finite set of values.

In fact if $Y_1 = [0,1]$

$$Y_{n,2m-1} = \text{support } y^+_{2^n+m}$$

$$Y_{n,2m} = \text{support } y^-_{2^n+m}$$

then for every λ the set of points t where $\Sigma\lambda_i d_i(t) = \lambda$ is a disjoint union of sets $D_{n,m}$ while the set of points r where $\Sigma\lambda_i y_i(t) = \lambda$ is the corresponding union of sets $Y_{n,m}$.

Our inequalities imply easily that

$$K_1 \leqq \frac{\mu(D_{n,m})}{\mu(Y_{n,m})} \leqq K_2$$

and

$$K_1 \leqq \frac{\mu(D_1)}{\mu(Y_1)} \leqq K_2$$

so that it readily follows that T is a lattice isomorphism with given estimates on norms.

The fact that D is complemented with projection of norm 1 follows from a theorem of Ando [1] which says that this is true for all lattice subspaces of $L_p[0,1]$ if $1 < p < \infty$.

We are now ready to state and prove:

THEOREM. *Let $\{x_i\}$ be a subsequence of the Haar system. Then if $1 < p < \infty$ and X is the closed linear span of $\{x_i\}$ in $L_p[0,1]$, either*

$$X \text{ is isomorphic to } l_p$$

or

$$X \text{ is isomorphic to } L_p[0,1].$$

PROOF. Consider the set

$$A = \{t \in [0,1] \mid t \in \text{support } x_i \text{ for infinitely many indices } i\}.$$

The set A is clearly measurable.

We show that:

$$\text{if } \mu(A) = 0 \text{ then } X \sim l_p$$

$$\text{if } \mu(A) > 0 \text{ then } X \sim L_p[0,1],$$

where $\sim$ means, as usual, isomorphic.

Case 1. $\mu(A) = 0$. We will show that X is a complemented subspace of a space isometric to l_p and appeal to Lemma 3. (X is clearly infinite-dimensional since it arises from a subsequence (infinite) of a basic sequence.)

First it is clear that if $\{S_n\}$ is a sequence of mutually disjoint measurable sets with $\mu([0,1] - \cup S_n) = 0$ and $X \mid S_n$ is the restriction of X to the set S_n then $(\Sigma X \mid S_n)_p$ is a subspace of $L_p[0,1]$ which contains X. The space X is clearly complemented in any such space by restriction of the projection of $L_p[0,1]$ onto X given in Lemma 1.

We will now choose sets $\{S_n\}$ with the above properties and so that $(\Sigma X \mid S_n)_p$ will obviously be isometric to l_p. In fact we will pick the sets so that X is constant on each one.

Consider the sets A_n, $n = 0, 1, \cdots,$:

$$A_n = \{t \in [0,1] \mid t \in \text{ support } x_i \text{ for exactly } n \text{ indices } i\}.$$

We mean, of course, the supports as given by the original definition of the Haar functions and not the support of a function equal a.e. to a Haar function.

The sets A and A_n are clearly pairwise disjoint and $[0,1] - \cup A_n = A$ so this set has measure zero.

Fix n for a moment and consider only those functions x_i which appear in the definition of A_n. By a maximality argument repeated n times, we can split this (possibly finite) sequence into n subsequences $\{x_i\}$ $i \in B_{n,j}$, $j = 1 \cdots n$ with each subsequence consisting of disjoint functions and so that the supports get finer, that is:

if $i \in B_{n,j}$ for $j \geqq 2$ then there exists $k \in B_{n,j-1}$ with supp $x_k \supset$ supp x_i (properly).

It is then easy to see that $A_n = \bigcup_{k \in B_{n,n}} (A_n \cap \operatorname{supp} x_k) = \bigcup_{k \in B_{n,n}} (A_n \cap \operatorname{supp} x_k^+) \cup \bigcup_{k \in B_{n,n}} (A_n \cap \operatorname{supp} s_k^-)$. The sets in the final union clearly are disjoint and have union A_n, and X is constant on each one.

The case $n = 0$ is easy since X is constant 0 on A_0.

Doing this for all n and numbering our sets properly, we arrive at the promised sets S_n; X is constant on each set S_n so $X \mid S_n$ is either 0 or 1-dimensional for all n.

Case 2. $\mu(A) > 0$. By Lemma 1, X is complemented in $L_p[0,1]$.

Suppose we can show that X contains a complemented subspace Y isomorphic to $L_p[0,1]$.

We will then get $X \sim L_p[0,1]$ by the decomposition method of Pełczyński.

We now proceed to construct such a subspace Y. We will pick for Y a block basic sequence with respect to $\{x_i\}$ sufficiently close to a sequence which satisfies the hypotheses of Lemma 4.

Consider the sequence $\{x_i\}$. By induction and a maximality argument we can find a countable number of subsequences $\{x_i\}$, $i \in N_k$ $(k = 1 \cdots)$, each possibly finite or infinite, with

$$N_k \cap N_j = \varnothing \text{ if } k \neq j$$

$$\cup N_k = N$$

if $i, j \in N_k$ for some k, then supp $x_i \cap \operatorname{supp} x_j = \varnothing$

unless $i = j$

if $i \in N_k$ for some $k \geqq 2$, then there exists

$j \in N_{k-1}$ with supp $x_i \subset$ supp x_j.

What we have really done is split up the sequence into a countable number of subsequences so that supports of functions get finer from one subsequence to the next and so that the functions in any one subsequence are disjoint from one another. This is similar to what we did in the previous case on the A_n. From the fact that $\mu(A) > 0$, it is easy to see from what follows that we actually have an infinite number of subsequences.

It is not difficult to see that the sets $A_k = \bigcup_{i \in N_k} \text{supp}\, x_i$ are decreasing, contain A, and $\bigcap_{k=1}^{\infty} A_k = A$.

For $n = -1, 0, \cdots$ and given positive constants c_n, pick k_n increasingly strictly with n so that

$$\mu(A_{k_n} - A) < c_n.$$

Then define functions b_i, $i = 1, 2, \cdots$ as follows:

$$b_1 = \sum_{i \in N_{k_{-1}}} x_i, \quad b_2 = \sum_{i \in N_{k_0}} x_i$$

and inductively thereafter by

$$b_{2^n+m} = \Sigma x_i, \text{ for } i \in N_{k_n}, \text{ supp } x_i \subset \text{supp } (b^+_{2^{n-1}+[(m+1)/2]})$$

if m is odd and

$$b_{2^n+m} = \Sigma x_i, \text{ for } i \in N_{k_n}, \text{ supp } x_i \subset \text{supp } (b^-_{2^{n-1}+[(m+1)/2]})$$

if m is even, for $n = 1 \cdots$ and $m = 1 \cdots 2^n$.

There is, of course, no problem with convergence since in all cases we are taking disjoint sums of functions with absolute value 1.

It is easy to see that the sequence $\{b_i\}$ does not necessarily satisfy the hypotheses of Lemma 4.

However, let $d_1 = |b_1| \,\big|_A$ and $d_i = b_i \,\big|_A$ for $i \geqq 2$.

We will show that if the constants c_n are picked small enough then Lemma 4 can be applied to $\{d_i\}$, and that there is a set I of integers for which $\Sigma_{i \in I} \| d_i' \| \, \| d_i - b_i \|$ is small enough to imply that $Y = [b_i]_{i \in I}$ is isomorphic to $L_p[0,1]$ and complemented in X. (It is clear that any such space Y is a subspace of X.)

If

$$B_1 = \text{support } b_1$$

$$B_{n,2m-1} = \text{support } b^+_{2^n+m}$$

$$B_{n,2m} = \text{support } b^-_{2^n+m}$$

it is then easy to see that

$$D_1 = A = B_1 \cap A$$

$$D_{n,m} = A \cap B_{n,m} \text{ where the sets}$$

D_1, $D_{n,m}$ correspond to $\{d_i\}$ as in Lemma 4.

Now all the functions b_i are symmetric for $i \geqq 2$ so that

$$\mu(B_{n,2m-1}) = \mu(B_{n,2m}) = \tfrac{1}{2}\mu(\text{support } b_{2^n+m}).$$

We get

$$\begin{aligned} \mu(B_{n,2m-1}) &\geqq \mu(D_{n,2m-1}) \\ &= \mu(B_{n,2m-1}) - \mu(B_{n,2m-1} - A) \\ &= \tfrac{1}{2}\mu(\text{support } b_{2^n+m}) - \mu(B_{n,2m-1} - A) \\ &\geqq \tfrac{1}{2}\mu(\text{support } b_{2^n+m}) - c_n \\ &\geqq \tfrac{1}{2}\mu(\text{support } d_{2^n+m}) - c_n \end{aligned}$$

since $B_{n,2m-1} \subset A_{k_n}$.

Similarly

$$\begin{aligned} \mu(B_{n,2m}) &\geqq \mu(D_{n,2m}) \geqq \tfrac{1}{2}\mu(\text{support } b_{2^n+m}) - c_n \\ &\geqq \tfrac{1}{2}\mu(\text{support } d_{2^n+m}) - c_n. \end{aligned}$$

On the other hand

$$\text{if } m \text{ is odd, support } d_{2^n+m} = \text{support } d^{+}_{2^{n-1}+[(m+1)/2]}$$

and so

$$\begin{aligned} \mu(\text{support } d_{2^n+m}) &= \mu(D_{n-1,2[(m+1)/2]-1}) \\ &= \mu(B_{n-1,2[(m+1)/2]-1}) - \mu(B_{n-1,2[(m+1)/2]-1} - A) \\ &= \tfrac{1}{2}\mu(\text{support}\, b_{2^{n-1}+[(m+1)/2]}) - \mu(B_{n-1,2[(m+1)/2]-1} - A) \\ &\geqq \tfrac{1}{2}\mu(\text{support}\, b_{2^{n-1}+[(m+1)/2]}) - c_{n-1} \\ &\geqq \tfrac{1}{2}\mu(\text{support}\, d_{2^{n-1}+[(m+1)/2]}) - c_{n-1}. \end{aligned}$$

If m is even, we get similarly

$$\mu(\text{support } d_{2^n+m}) \geqq \mu(\text{support } d_{2^{n-1}+[(m+1)/2]}) - c_{n-1}.$$

Combining these inequalities we get

$$\mu(D_{n,m}) \geqq \tfrac{1}{2}\mu(\text{support}\, d_{2^n+m}) - c_n$$
$$\geqq \tfrac{1}{4}\mu(\text{support}\, d_{2^{n-1}+[(m+1)/2)]}) - \tfrac{1}{2}c_{n-1} - c_n$$
$$\geqq \tfrac{1}{8}\mu\,(\text{support}\, d_{2^{n-2}+[([(m+1)/2]+1)/2]}) - \tfrac{1}{4}c_{n-2} - \tfrac{1}{2}c_{n-1} - c_n \geqq \cdots$$

and hence inductively

$$\mu(D_{n,m}) \geqq \frac{1}{2^{n+1}}\ \mu(\text{support}\, b_2) - \sum_{j=0}^{n} c_j 2^{j-n}$$

and so

$$2^{n+1}\mu(D_{n,m}) \geqq \mu(\text{support}\, b_2) - \sum_{j=0}^{n} c_j 2^{j+1}$$
$$\geqq \mu(A) - \sum_{j=0}^{\infty} c_j 2^{j+1}.$$

Thus, our first requirement on the constants c_j is that

$$K_1 = \mu(A) - \sum_{j=0}^{\infty} c_j 2^{j+1} > 0.$$

This can easily be done since $\mu(A) > 0$.

It is easy to see that

$$\mu(D_1) \geqq \mu(B_1) \geqq \mu(A) \geqq K_1.$$

We can show on the other hand that

$$\mu(D_{n,2m}) \leqq \mu(B_{n,2m}) = \tfrac{1}{2}\mu(\text{support}\, b_{2^n+m})$$
$$\leqq \tfrac{1}{2}\mu(\text{support}\, b^{+}_{2^{n-1}+[(m+1)/2]})$$
$$= \tfrac{1}{4}\mu(\text{support}\, b_{2^{n-1}+[(m+1)/2]})$$
$$\leqq \tfrac{1}{8}\mu(\text{support}\, b^{\pm}_{2^{n-2}+k})$$
$$\leqq \cdots \leqq \frac{1}{2^{n+1}}\mu(\text{support}\, b_2)$$

where m is odd and the $\pm$ is either $+$ or $-$ depending on m, and k depends on m.

A similar argument implies that

$$\mu(D_{n,m}) \leqq \frac{1}{2^{n+1}}\mu(\text{support}\, b_2) \text{ for all } n, m.$$

But $\mu(D_1) \leqq \mu(B_1) \leqq \mu(\text{support}\, b_2) \leqq \mu(A) + c_0 \leqq \mu(A) + c_{-1}$ and so

$$2^{n+1}\mu(D_{n,m}) \leqq \mu(A) + c_{-1}$$
$$\mu(D_1) \leqq \mu(A) + c_{-1}.$$

We let $K_2 = \mu(A) + c_{-1}$ and apply Lemma 4. Then D is isomorphic to $L_p[0,1]$ and complemented by a projection of norm 1.

Consider the index set I consisting of 2 and all integers of the form $2^n + m$ for $n = 1 \cdots$ and $m = 1 \cdots 2^{n-1}$.

By Lemma 1 there is a constant K depending only on the Haar system such that $[y_i]_{i \in I}$ is complemented by a projection of norm at most K.

But then $[d_i]_{i \in I}$ is complemented in D, hence also in $L_p[0,1]$, by a projection with norm at most $K \| T \| \| T^{-1} \|$, where $T(d_i) = y_i$. (D is complemented, norm 1, in $L_p[0,1]$.)

If we can show that

$$W = \sum_{i \in I} K \| T \| \| T^{-1} \| \| d_i' \| \| d_i - b_i \| < 1 \text{ then, by Lemma 2,}$$

$Y = [b_i]_{i \in I}$ will be complemented in $L_p[0,1]$ and isomorphic to $[d_i]_{i \in I}$ and hence isomorphic to $[y_i]_{i \in I}$.

But $y_i' T(d_j) = y_i'(y_i) = \delta_{ij}$ so that $y_i' T = d_i'$. The functional y_i' consists in integrating with respect to y_i, with a normalizing factor.

It is not difficult to show that $\| y'_{2^n+m} \| \leqq 2^{n/p}$.

Thus

$$\| d'_{2^n+m} \| \leqq \| y'_{2^n+m} \| \| T \| \leqq 2^{n/p} \| T \|.$$

Secondly

$$\| d_{2^n+m} - b_{2^n+m} \| = (\mu(\text{support } b_{2^n+m} - A))^{1/p}$$

and so

$$\leqq c_n^{1/p}$$

$$W \leqq K \| T \| \| T^{-1} \| \left(\| d_2' \| \| d_2 - b_2 \| + \sum_{n=1}^{\infty} \sum_{m=1}^{2^{n-1}} \| d'_{2^n+m} \| \| d_{2^n+m} - b_{2^n+m} \| \right)$$

$$\leqq K \| T \| \| T^{-1} \| \left(\| T \| c_0^{1/p} + \sum_{n=1}^{\infty} 2^{n-1} 2^{n/p} \| T \| c_n^{1/p} \right)$$

$$\leqq K \left[\frac{1}{\mu(A) - \sum_0^{\infty} c_j 2^{j+1}} \right]^{2/p} (\mu(A) + c_{-1})^{1/p} \left(c_0^{1/p} + \sum_1^{\infty} c_n^{1/p} 2^{n-1+n/p} \right)$$

and this can be made smaller than 1 at the same time as getting $K_1 > 0$ if

$$\left(\frac{\mu(A) + c_{-1}}{\mu(A) - \sum c_j 2^{j+1}} \right)^{1/p} \leqq 2$$

and

$$K\,\frac{c_0^{1/p}+\sum c_n^{1/p}2^{n-1+n/p}}{(\mu(A)-\sum c_j 2^{j+1})^{1/p}} \leqq \frac{1}{4}.$$

This is easily done.

Thus $[b_i]_{i\in I}$ is isomorphic to $[y_i]_{i\in I}$ and complemented in $L_p[0,1]$ hence also in X. But by unconditionality of the Haar system, $[y_i]_{i\in I}$ is clearly isomorphic to $L_p[0,\frac{1}{2}]$ and so also to $L_p[0,1]$.

QUESTION. What are the isomorphic types of complemented closed linear spans of blocks with respect to the Haar system? If $1<p<\infty$, is every separable $\mathscr{L}_p$-space thus realizable? How about $\mathscr{L}_p$-spaces with unconditional basis?

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